Position-Required and its Application to Guidance Problems

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This paper presents and develops a concept that, compared with previous methods, will greatly improve the computational feasibility of second-order approximations to velocity-required for a wide variety of boundary-value problems arising in guidance. First, it is shown that a large category of boundary-value problems and trajectory shaping constraints may be included within a variable time-of-arrival constraint and a variable target position constraint. This observation is of paramount importance since it makes guidance equations much more explicit and may avoid extensive preflight shaping of powered flight trajectories. Then the concept of position-required is developed, and Riccati-type differential equations are derived which the first- and second-order partial derivatives of position-required obey. The number of such equations is quite small (between 33 and 87 depending upon the particular boundary-value problem) compared with 168 equations to obtain all of the first- and second-order miss (sensitivity) coefficients. A crucial property of the differential equations for the derivatives of position-required is that initial conditions at the target are known a priori, which is in contrast with the lack of initial conditions for similar equations derived previously for velocity-required. In a region where velocity-required and its derivatives exist, algebraic equations are derived from which the derivatives of velocity-required can be calculated from the derivatives of position-required.

Introduction

LET us consider an object whose equations of ballistic motion are

$$dx/dt = \dot{x}, \; d\dot{x}/dt = g(x) + a(x,\dot{x}) \tag{1}$$

where x, $\dot{x}$, g, and a are the vectors of position, velocity, and gravitational and aerodynamic acceleration, respectively. We assume that the vectors g and a possess continuous second-order derivatives. Although these equations are frequently expressed in an inertial coordinate system, there is nothing in the subsequent derivations to preclude a rotating coordinate system. Merely include in the vector a the centripetal and Coriolis accelerations. Let the solutions to Eq. (1) at some time t be denoted by $x(x_0,\dot{x}_0,t_0,t)$ and $\dot{x}(x_0,\dot{x}_0,t_0,t)$ for initial conditions $(x_0,\dot{x}_0,t_0)$. Now suppose we wish to hit a target at some time t_I where the position vector of the target at this time is denoted by $x_T(t_I)$. Hence, this requirement manifests itself in the equation

$$x_T(t_I) = x(x_0,\dot{x}_0,t_0,t_I) \tag{2}$$

If t_I is fixed at some value t_F, then we can usually solve Eq. (2) for $\dot{x}_0$ as a function of x_0 and t_0, called the velocity-required (correlated velocity) vector v_r for the constant total time-of-flight constraint, such that $x_T(t_F) = x[x_0,v_r(x_0,t_0),t_0,t_F]$.

If Eq. (1) is sufficiently complicated by oblate gravitational and aerodynamic accelerations, explicit solutions for v_r are not known, and we must resort to approximations. A computationally efficient and accurate method to obtain a linear approximation to v_r is to solve the first-order variational (linearized, perturbation) equations of Eq. (1), obtaining linear sensitivity coefficients, and then solve for $\partial v_r/\partial x_0$ by

$$\partial v_r/\partial x_0 = -(\partial x/\partial \dot{x}_0)^{-1}(\partial x/\partial x_0) \tag{3}$$

with a similar equation for $\partial v_r/\partial t_0$.[1]

There are two current methods of obtaining better-than-linear approximations to desired solutions. First, a finite number of discrete points are chosen at which v_r is calculated accurately by means of an iteration procedure involving repetitive solutions of Eq. (1). The coefficients of appropriately chosen polynomials are then discrete least-mean-square-fitted to the accurate v_r data (Ref. 2, pp. 228–230). The second method is to solve Eq. (1) and its first- and second-order variational equations to obtain first- and second-order sensitivity coefficients. These coefficients can then be used to calculate $\partial v_r/\partial x_0$, as in Eq. (3), and $\partial^2 v_r/\partial x_0^2$ from a similar equation.[3] The first method is the most widely used technique even though repetitively solving Eq. (1) can be time consuming. The second method suffers from the fact that 168 differential equations would have to be integrated [including Eq. (1), its first-order variational equations, and the reduction owing to the equality of some mixed second-order partial derivatives]. For the constant total time-of-flight constraint, it is possible to derive differential equations that $\partial v_r/\partial x_0$ and $\partial^2 v_r/\partial x_0^2$ obey directly so that only 33 differential equations have to be integrated, including Eq. (1) [Ref. 4, Eq. (6.57), and Ref. 5]. These differential equations require initial conditions, and the only way we know to obtain them is to solve the first- and second-order variational equations of Eq. (1) and to use Eq. (3) and something similar for $\partial^2 v_r/\partial x_0^2$. However, we wish to avoid solving the second-order variational equations.

Now it is known that $(\partial v_r/\partial x_0)^{-1}$ is zero at the target, and a differential equation can be derived that it obeys [Ref. 4, Eq. (6.59)]. The extension of this idea to second-order approximations is facilitated by the introduction of a new concept that gives the proper physical interpretation to $(\partial v_r/\partial x_0)^{-1}$. Observe from Eq. (3) that $(\partial v_r/\partial x_0)^{-1} = -(\partial x/\partial x_0)^{-1}(\partial x/\partial \dot{x}_0)$, which is the same as solving Eq. (2) for position as a function of velocity rather than the usual solution of velocity as a function of position. Thus arises the concept of position-required p_r. Hence, the basic idea (an idea of wide application apparently unarticulated heretofore) is to solve a

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boundary-value problem (in particular, position-required) that does not have a singularity at the end condition (target) and then invert this solution to the desired one (namely, velocity-required), which has a singularity at the end condition, in a region where inversion is possible.

To motivate a sufficiently general derivation of position-required, its derivatives, and the differential equations which they obey, we make the important observation that many mission and physical constraints may be included in the velocity-required function by suitably varying the time-of-arrival t_I at the target. Such an inclusion eliminates the need for trajectory shaping in many cases since all the constraints may be built into the velocity-required function. The result is a much more explicit set of guidance equations, and we will discuss this aspect of guidance first.

Trajectory Shaping and Time-of-Arrival Variation

Fixing t_I may not be a real mission requirement, and there may be auxiliary constraints on the mission and physical configurations in addition to hitting the target. Since we must hit the target, Eq. (2) shows that the only variable that we can vary freely and upon which v_r depends directly is t_I. Therefore, we wish to solve Eq. (2) for v_r as a function of x_0, t_0, and t_I such that $x_T(t_I) = x[x_0,v_r(x_0,t_0,t_I),t_0,t_I]$. The importance of a solution of this kind is that we are free to vary the time-of-arrival t_I to satisfy additional constraints on the mission and physical configuration.† Note that even though we vary t_I, the function $v_r(x_0,t_0,t_I)$ guarantees we shall hit the target at whatever t_I we choose. Let us discuss some possible auxiliary constraints.

Minimum fuel and minimum burn time are both related to the velocity-to-be-gained vector v_g, defined by $v_g \equiv v_r - v$ where v is the current velocity of the booster, and the current position and time of the booster are used to calculate v_r. It is now clear that the current value of v_g depends upon t_I so t_I can be chosen to minimize the magnitude of v_g (Ref. 4, pp. 267–271).

Another example is to avoid collision with some other object, such as a satellite, whose position vector as a function of time t' is known as $y(t')$. In this situation, we require that $|y(t') - x[x_0,v_r(x_0,t_0,t_I),t_0,t']| \geq K_s$. The constant K_s is some number indicating minimum allowable spacing. It may then be possible to solve this inequality (or more appropriately its square) for t_I to satisfy this constraint. Finally, a value of t_I might then be found that minimizes the magnitude of v_g subject to the inequality constraint.

A constraint on the physical configuration may be that the commanded yaw attitude ψ_c in steering does not exceed a certain limit, for example K_y, to avoid ripping cables connecting the power source fixed to the vehicle body and the inertial measurement unit. For simplicity, let us assume that our steering law points the thrust vector along the velocity-to-be-gained vector; and, for many important applications, this is quite true. As before, v_g depends upon t_I and so then does the direction cosine vector ξ of v_g; and, since the relationship between ξ and ψ_c is known (frequently one component such as ξ_2 is of the form $\xi_2 = -\sin\psi_c$), it may be possible to solve for t_I so that $|\psi_c| \leq K_y$.

Although there are many more examples, these few serve to illustrate the power of the idea of varying time-of-arrival to satisfy auxiliary constraints. If auxiliary equations must be satisfied, they or their approximations also will need to be calculated. Such additional calculations are not discussed in this paper. We restrict ourselves to obtaining approximations to v_r as a function of x_0,t_0,t_I, and x_I (which will be defined below).

In Eq. (2), we have assumed that $x_T(t_I)$ is known. However, there are cases where $x_T(t_I)$ may not be known accurately owing to inaccurate or incomplete tracking, or we may wish to vary the target site for other reasons. In these situations, we would like to know v_r as a function of an arbitrary impact position x_I, as well as x_0,t_0, and t_I.

In summary, a very large category of realistic guidance problems can be efficiently handled by knowing v_r as a function of x_I,x_0,t_0, and t_I and by suitably varying the time-of-arrival t_I. It is often time consuming and expensive to shape, by various iterative methods, powered flight trajectories to satisfy these constraints. However, this idea illustrates how one may be able to shape during powered flight by varying the time-of-arrival appropriately during each guidance computation cycle. Surely this results in a more explicit set of guidance equations. Most importantly, we note that this concept of using variable time-of-arrival and target position in the velocity-required expression to shape powered flight trajectories is quite independent of the remainder of this paper and has been utilized in applications which use only linear sensitivity coefficients.

† The variable time-of-arrival constraint, where t_I is chosen to satisfy additional constraints, is to be distinguished from the nonconstant total time-of-flight constraint where t_I is of no interest [Ref. 2, Eqs. (5.71) and (5.72)]. We remark that the derivations of position-required, its derivatives, the differential equations which they obey, and the relationships to velocity-required and its derivatives have been extended to the nonconstant total time-of-flight constraint.[6]

First-Order Derivatives of Position-Required

In this section, we derive differential equations that $\partial p_r/\partial \dot{x}_0$, $\partial p_r/\partial x_I$, and $\partial p_r/\partial t_I$ obey and determine the initial conditions for these differential equations at the target [Eqs. (10, 13, and 16)]. We also show that it is not necessary to solve the differential equation for $\partial p_r/\partial t_I$ [Eq. (16)] since there is an algebraic relationship between $\partial p_r/\partial x_I$ and $\partial p_r/\partial t_I$ [Eq. (17)]. In the next section, we derive differential equations and their initial conditions for the second-order derivatives of position-required, and in the final section we derive relationships between the derivatives of velocity-required and position-required.

Suppose we have a specific set of initial conditions $(x_0',\dot{x}_0',t_0')$ and at some time-of-flight t_I' we define $x_I' \equiv x(x_0',\dot{x}_0',t_0',t_I')$. We wish to solve for v_r as a function of x_0,t_0,t_I, and x_I such that $x_I = x[x_0,v_r(x_0,t_0,t_I,x_I),t_0,t_I]$ for any (x_0,t_0,t_I,x_I) in a neighborhood of (x_0',t_0',t_I',x_I'). This is our general boundary-value problem, and x_I and t_I may be varied for a variety of reasons mentioned previously. To derive the desired results efficiently in the second-order case, we consider Eq. (1) as being solved backwards from the target, and we need the crucial property that, for initial conditions $(x_I,\dot{x}_I,t_I)$ close to $(x_I',\dot{x}_I',t_I')$, where $\dot{x}_I' \equiv \dot{x}(x_0',\dot{x}_0',t_0',t_I')$,

$$x(x_I,\dot{x}_I,t_I,t) = p_r[\dot{x}(x_I,\dot{x}_I,t_I,t),t,t_I,x_I] \tag{4}$$

is true for any time t such that the matrix $\partial x/\partial x_0$ from t to t_I' along the reference trajectory defined by $(x_0',\dot{x}_0',t_0')$ is nonsingular. This matrix certainly is nonsingular for $t = t_I'$ since it is the (3×3) identity matrix there. What Eq. (4) says is that if we fly a trajectory backwards from the target at time t_I, then the position at time t on this trajectory is precisely the position-required to hit the target starting at time t and flying in the forward direction to time t_I. A simple argument involving the uniqueness and continuity of solutions of Eq. (1) with respect to initial conditions and the uniqueness of p_r will prove this (see the Appendix for a rigorous derivation). As mentioned we are now considering Eq. (1) as being solved backwards from the target with initial conditions $(x_I,\dot{x}_I,t_I)$. Hence, position, velocity and time back along the trajectory henceforth will be denoted by x, $\dot{x}$, and t rather than x_0, $\dot{x}_0$, and t_0, and $\partial p_r/\partial \dot{x}_0$ now becomes $\partial p_r/\partial \dot{x}$.

It follows from Eqs. (1) and (4) that the total derivative of p_r with respect to t is

$$dp_r/dt = \dot{x} \tag{5}$$

Now p_r is a function of $\dot{x}_I$ and is continuously differentiable with respect to $\dot{x}_I$ provided the right-hand side of Eq. (5) is continuously differentiable with respect to p_r and $\dot{x}_I$. Since $\dot{x}$ does not depend upon p_r, it is continuously differentiable with respect to it, and $\dot{x}$ is continuously differentiable with respect to $\dot{x}_I$ from basic variational theory (Ref. 7, pp. 25–32).‡ Hence,

$$(d/dt)\left(\frac{\partial p_r}{\partial \dot{x}_I}\right) = \partial \dot{x}/\partial \dot{x}_I \tag{6}$$

From Eq. (4) it is clear that

$$\partial x/\partial \dot{x}_I = (\partial p_r/\partial \dot{x})(\partial \dot{x}/\partial \dot{x}_I) \tag{7}$$

Now at $t = t_I'$, $\partial \dot{x}/\partial \dot{x}_I$ is the (3×3) identity matrix. We assume it is nonsingular for all t; but then the total derivative of $\partial \dot{x}/\partial \dot{x}_I$ and $\partial p_r/\partial \dot{x}$ with respect to t exists since $\partial p_r/\partial \dot{x} = (\partial x/\partial \dot{x}_I)(\partial \dot{x}/\partial \dot{x}_I)^{-1}$ [see Eq. (7)], and these matrices, in turn, are differentiable with respect to t. Thus, Eq. (6) may be expanded to

$$[(d/dt)(\partial p_r/\partial \dot{x})]\partial \dot{x}/\partial \dot{x}_I + (\partial p_r/\partial \dot{x})[(d/dt)(\partial \dot{x}/\partial \dot{x}_I)] = \partial \dot{x}/\partial \dot{x}_I \tag{8}$$

using $\partial p_r/\partial \dot{x}_I = (\partial p_r/\partial \dot{x})(\partial \dot{x}/\partial \dot{x}_I)$, which comes from using the chain rule on the right-hand side of Eq. (4). Recall from variational theory that the first-order variational equations of Eq. (1) are

$$\frac{d}{dt}\begin{pmatrix} \frac{\partial x}{\partial x_I} & \frac{\partial x}{\partial \dot{x}_I} & \frac{\partial x}{\partial t_I} \\ \frac{\partial \dot{x}}{\partial x_I} & \frac{\partial \dot{x}}{\partial \dot{x}_I} & \frac{\partial \dot{x}}{\partial t_I} \end{pmatrix} = \begin{pmatrix} 0 & I \\ \frac{\partial g}{\partial x} + \frac{\partial a}{\partial x} & \frac{\partial a}{\partial \dot{x}} \end{pmatrix} \begin{pmatrix} \frac{\partial x}{\partial x_I} & \frac{\partial x}{\partial \dot{x}_I} & \frac{\partial x}{\partial t_I} \\ \frac{\partial \dot{x}}{\partial x_I} & \frac{\partial \dot{x}}{\partial \dot{x}_I} & \frac{\partial \dot{x}}{\partial t_I} \end{pmatrix} \tag{9}$$

where 0 and I denote the (3×3) zero and identity matrices, respectively. The initial conditions at $t = t_I'$ for Eq. (9) are

$$\begin{pmatrix} I & 0 & -\dot{x}_I' \\ 0 & I & -g(x_I') - a(x_I', \dot{x}_I') \end{pmatrix}$$

We then substitute from Eq. (9) into Eq. (8) obtaining

$$\left[\frac{d}{dt}\left(\frac{\partial p_r}{\partial \dot{x}}\right)\right]\frac{\partial \dot{x}}{\partial \dot{x}_I} + \frac{\partial p_r}{\partial \dot{x}}\left[\left(\frac{\partial g}{\partial x} + \frac{\partial a}{\partial x}\right)\frac{\partial x}{\partial \dot{x}_I} + \frac{\partial a}{\partial \dot{x}}\frac{\partial \dot{x}}{\partial \dot{x}_I}\right] = \frac{\partial \dot{x}}{\partial \dot{x}_I}$$

which, using Eq. (7), becomes

$$\left[\frac{d}{dt}\left(\frac{\partial p_r}{\partial \dot{x}}\right) + \frac{\partial p_r}{\partial \dot{x}}\left(\frac{\partial g}{\partial x} + \frac{\partial a}{\partial x}\right)\frac{\partial p_r}{\partial \dot{x}} + \frac{\partial p_r}{\partial \dot{x}}\frac{\partial a}{\partial \dot{x}} - I\right]\frac{\partial \dot{x}}{\partial \dot{x}_I} = 0$$

Because of our assumption that $\partial \dot{x}/\partial \dot{x}_I$ is nonsingular, this must reduce to

$$\frac{d}{dt}\left(\frac{\partial p_r}{\partial \dot{x}}\right) + \frac{\partial p_r}{\partial \dot{x}}\left(\frac{\partial g}{\partial x} + \frac{\partial a}{\partial x}\right)\frac{\partial p_r}{\partial \dot{x}} + \frac{\partial p_r}{\partial \dot{x}}\frac{\partial a}{\partial \dot{x}} = I \tag{10}$$

Equation (7) and the initial conditions for Eq. (9) show that the value of $\partial p_r/\partial \dot{x}$ at $t = t_I'$ is zero. Notice that, since $\partial p_r/\partial \dot{x} = (\partial v_r/\partial x)^{-1}$, Eq. (10) (with the aerodynamic terms omitted) is the same as Eq. (6.59) of Ref. 4.

Now let us return to Eq. (5). As before, the total derivative of p_r with respect to x_I, denoted by dp_r/dx_I, satisfies

$$(d/dt)(dp_r/dx_I) = \partial \dot{x}/\partial x_I \tag{11}$$

But $dp_r/dx_I = (\partial p_r/\partial \dot{x})(\partial \dot{x}/\partial x_I) + \partial p_r/\partial x_I$, again using the chain rule on the right-hand side of Eq. (4), so that we may then expand Eq. (11) to

$$\left[\frac{d}{dt}\left(\frac{\partial p_r}{\partial \dot{x}}\right)\right]\frac{\partial \dot{x}}{\partial x_I} + \frac{\partial p_r}{\partial \dot{x}}\left[\frac{d}{dt}\left(\frac{\partial \dot{x}}{\partial x_I}\right)\right] + \frac{d}{dt}\left(\frac{\partial p_r}{\partial x_I}\right) = \frac{\partial \dot{x}}{\partial x_I}$$

Substituting from Eq. (9), noting from Eq. (4) that

$$\partial x/\partial x_I = (\partial p_r/\partial \dot{x})(\partial \dot{x}/\partial x_I) + \partial p_r/\partial x_I \tag{12}$$

and combining terms yield

$$\left[\frac{d}{dt}\left(\frac{\partial p_r}{\partial \dot{x}}\right) + \frac{\partial p_r}{\partial \dot{x}}\left(\frac{\partial g}{\partial x} + \frac{\partial a}{\partial x}\right)\frac{\partial p_r}{\partial \dot{x}} + \frac{\partial p_r}{\partial \dot{x}}\frac{\partial a}{\partial \dot{x}} - I\right]\frac{\partial \dot{x}}{\partial x_I} + \frac{d}{dt}\left(\frac{\partial p_r}{\partial x_I}\right) + \frac{\partial p_r}{\partial \dot{x}}\left(\frac{\partial g}{\partial x} + \frac{\partial a}{\partial x}\right)\frac{\partial p_r}{\partial x_I} = 0$$

Thus, using Eq. (10) we have

$$(d/dt)(\partial p_r/\partial x_I) + (\partial p_r/\partial \dot{x})(\partial g/\partial x + \partial a/\partial x)\partial p_r/\partial x_I = 0 \tag{13}$$

The initial condition for Eq. (13) at $t = t_I'$ is the (3×3) identity matrix as shown by evaluating Eq. (12) at t_I' and using the initial conditions for Eq. (9). Note that Eq. (13) is a linear differential equation in the matrix variable $\partial p_r/\partial x_I$. In fact, $\partial p_r/\partial x_I$ is the fundamental matrix of the linear vector differential equation

$$dy/dt + (\partial p_r/\partial \dot{x})(\partial g/\partial x + \partial a/\partial x)y = 0 \tag{14}$$

Let us return again to Eq. (5). Now the total derivative of p_r with respect to t_I, denoted by dp_r/dt_I, satisfies

$$(d/dt)(dp_r/dt_I) = \partial \dot{x}/\partial t_I$$

But then $dp_r/dt_I = (\partial p_r/\partial \dot{x})(\partial \dot{x}/\partial t_I) + \partial p_r/\partial t_I$, and noting that

$$\partial x/\partial t_I = (\partial p_r/\partial \dot{x})(\partial \dot{x}/\partial t_I) + \partial p_r/\partial t_I \tag{15}$$

we proceed as before to obtain

$$\left[\frac{d}{dt}\left(\frac{\partial p_r}{\partial \dot{x}}\right) + \frac{\partial p_r}{\partial \dot{x}}\left(\frac{\partial g}{\partial x} + \frac{\partial a}{\partial x}\right)\frac{\partial p_r}{\partial \dot{x}} + \frac{\partial p_r}{\partial \dot{x}}\frac{\partial a}{\partial \dot{x}} - I\right]\frac{\partial \dot{x}}{\partial t_I} + \frac{d}{dt}\left(\frac{\partial p_r}{\partial t_I}\right) + \frac{\partial p_r}{\partial \dot{x}}\left(\frac{\partial g}{\partial x} + \frac{\partial a}{\partial x}\right)\frac{\partial p_r}{\partial t_I} = 0$$

Using Eq. (10), this becomes

$$(d/dt)(\partial p_r/\partial t_I) + (\partial p_r/\partial \dot{x})(\partial g/\partial x + \partial a/\partial x)\partial p_r/\partial t_I = 0 \tag{16}$$

Using Eq. (15) and the initial conditions for Eq. (9), we can see that the initial condition at t_I' for Eq. (16) is $-\dot{x}_I'$. However, since Eq. (16) is of identical form to Eq. (14), of which $\partial p_r/\partial x_I$ is the fundamental matrix, we know from variational theory that

$$\partial p_r/\partial t_I = -(\partial p_r/\partial x_I)\dot{x}_I' \tag{17}$$

for any time t. Hence, it is not necessary to solve Eq. (16). We may use Eq. (17) instead.

The simultaneous solution of 24 differential equations [Eqs. (1, 10, and 13)] gives $\partial p_r/\partial \dot{x}$ and $\partial p_r/\partial x_I$, and then $\partial p_r/\partial t_I$ algebraically [by means of Eq. (17)]. These coefficients allow us to compute linear estimates of p_r for arbitrary variations in $\dot{x}$, x_I, and t_I. However, not all problems involve arbitrary variations in all these variables. For example, the constant total time-of-flight constraint only allows variation in $\dot{x}$ so that only 15 differential equations have to be solved [Eqs. (1) and (10)]. Another important example is illustrated by varying the time-of-arrival t_I with the target moving so that the desired aim point is $x_I = x_T(t_I)$. In this instance, variations in $\dot{x}$ and t_I may be arbitrary but those in x_I are not. In this example, a linear approximation to the variation in p_r is given by

$$\Delta p_r \approx (\partial p_r/\partial \dot{x})\Delta \dot{x} + (\partial p_r/\partial x_I)[x_T(t_I) - x_T(t_I')] + (\partial p_r/\partial t_I)\Delta t_I \tag{18}$$

‡ It was R. Rinker's suggestion that led to this particular observation and which significantly reduces the algebraic manipulations necessary to derive the desired equations in the second-order situation. The original derivation, which used the Implicit Function Theorem, was much more lengthy.

However, $x_T(t_I) - x_T(t_I')$ may be linearly approximated by $[\dot{x}_T(t_I')]\Delta t_I$ so we have

$$\Delta p_r \approx (\partial p_r/\partial \dot{x})\Delta \dot{x} + [(\partial p_r/\partial x_I)\dot{x}_T(t_I') + \partial p_r/\partial t_I]\Delta t_I \quad (19)$$

Since $\partial p_r/\partial t_I = -(\partial p_r/\partial x_I)\dot{x}_I'$, it is obvious that the vector $(\partial p_r/\partial x_I)[\dot{x}_T(t_I')] + \partial p_r/\partial t_I$ is the solution of Eq. (14) with initial condition at the target $\dot{x}_T(t_I') - \dot{x}_I'$. In this example, only 18 differential equations must be solved simultaneously [(Eqs. (1, 10, and 14)]. Since 18 differential equations are only six less than 24, this reduction is not particularly important for linear approximations; however, the number of differential equations to obtain the appropriate second-order coefficients is considerably less than they are when variations in x_I are arbitrary.

Second-Order Derivatives of Position-Required

In this section we derive the differential equations which the second-order partial derivatives of position-required obey along a reference ballistic trajectory. The differential equations for $\partial^2 p_{ri}/\partial\dot{x}_j\partial\dot{x}_k$, $\partial^2 p_{ri}/\partial\dot{x}_j\partial x_{Ik}$, and $\partial^2 p_{ri}/\partial x_{Ij}x_{Ik}$, $i,j,k = 1,2,3$, are Eqs. (23, 25, and 27), respectively. The initial conditions at the target for these differential equations are all zero. Algebraic equations are derived for $\partial^2 p_{ri}/\partial\dot{x}_j\partial t_I$, $\partial^2 p_{ri}/\partial x_{Ij}\partial t_I$, and $\partial^2 p_{ri}/\partial t_I^2$, $i,j = 1,2,3$ [Eqs. (28, 29, and 30)]. Hence, to obtain all of the possible first- and second-order partial derivatives of p_r, it is necessary to solve 87 differential equations simultaneously [Eqs. (1, 10, 13, 23, 25, and 27)], taking into account the equality of some mixed second-order partial derivatives. For the constant total time-of-flight constraint, only 33 differential equations have to be solved [Eqs. (1, 10, and 23)]. Both of these numbers (87 and 33) are considerably less than the 168 differential equations required to calculate all of the first- and second-order sensitivity coefficients. In the important special case where variations in x_I are not arbitrary but are constrained to be $x_T(t_I)$ and where these variations are approximated by $[\dot{x}_T(t_I')]\Delta t_I$, only 48 differential equations have to be solved [Eqs. (1, 10, 14, 23, 32, and 33)] with the appropriate initial conditions.

It is not possible, nor necessarily desirable, to include the complete derivation of all of the desired equations in this paper. Thus only those steps are included which will enable the interested reader to reconstruct all the necessary steps with a moderate effort. Moreover, we adopt the Einstein summation convention which is so common in classical tensor analysis.[8] All indices and sums over indices in this section vary over 1, 2, 3.

We first consider the second-order partials of position-required with respect to velocity $\dot{x}$. To derive the desired equations, we write Eq. (10) in coordinate form as

$$\frac{d}{dt}\left(\frac{\partial p_{ri}}{\partial \dot{x}_k}\right) + \frac{\partial p_{ri}}{\partial \dot{x}_m}\left(\frac{\partial g_m}{\partial x_n} + \frac{\partial a_m}{\partial x_n}\right)\frac{\partial p_{rn}}{\partial \dot{x}_k} + \frac{\partial p_{ri}}{\partial \dot{x}_m}\frac{\partial a_m}{\partial \dot{x}_k} = \delta_{ik} \quad (20)$$

where δ_{ik} is the Kronecker delta function. Equation (20) is continuously differentiable with respect to $\dot{x}_I$ since the second-order derivatives of g and a are continuous. Expressions for these second-order derivatives for the dynamics presented in Ref. 1 have been derived.[9] Hence, applying variational theory to Eq. (20) we have

$$\frac{d}{dt}\left(\frac{\partial^2 p_{ri}}{\partial \dot{x}_{Il}\partial \dot{x}_k}\right) + \frac{\partial^2 p_{ri}}{\partial \dot{x}_{Il}\partial \dot{x}_m}\left(\frac{\partial g_m}{\partial x_n} + \frac{\partial a_m}{\partial x_n}\right)\frac{\partial p_{rn}}{\partial \dot{x}_k} + \frac{\partial p_{ri}}{\partial \dot{x}_m}\left(\frac{\partial^2 g_m}{\partial \dot{x}_{Il}\partial x_n} + \frac{\partial^2 a_m}{\partial \dot{x}_{Il}\partial x_n}\right)\frac{\partial p_{rn}}{\partial \dot{x}_k} + \frac{\partial p_{ri}}{\partial \dot{x}_m}\left(\frac{\partial g_m}{\partial x_n} + \frac{\partial a_m}{\partial x_n}\right)\frac{\partial^2 p_{rn}}{\partial \dot{x}_{Il}\partial \dot{x}_k} + \frac{\partial^2 p_{ri}}{\partial \dot{x}_{Il}\partial \dot{x}_m}\frac{\partial a_m}{\partial \dot{x}_k} + \frac{\partial p_{ri}}{\partial \dot{x}_m}\frac{\partial^2 a_m}{\partial \dot{x}_{Il}\partial \dot{x}_k} = 0 \quad (21)$$

Noting from the right-hand side of Eq. (4) that $\partial^2 p_{ri}/\partial\dot{x}_{Il}\partial\dot{x}_k = (\partial^2 p_{ri}/\partial\dot{x}_j\partial\dot{x}_k)(\partial\dot{x}_j/\partial\dot{x}_{Il})$, substituting from Eqs. (7) and (9), and interchanging some dummy summation indices, Eq. (21) expands to

$$\left[\frac{d}{dt}\left(\frac{\partial^2 p_{ri}}{\partial \dot{x}_j\partial \dot{x}_k}\right)\right]\frac{\partial \dot{x}_j}{\partial \dot{x}_{Il}} + \frac{\partial^2 p_{ri}}{\partial \dot{x}_j\partial \dot{x}_k}\left[\left(\frac{\partial g_j}{\partial x_s} + \frac{\partial a_j}{\partial x_s}\right)\frac{\partial p_{rs}}{\partial \dot{x}_n}\frac{\partial \dot{x}_n}{\partial \dot{x}_{Il}} + \frac{\partial a_j}{\partial \dot{x}_s}\frac{\partial \dot{x}_s}{\partial \dot{x}_{Il}}\right] + \frac{\partial^2 p_{ri}}{\partial \dot{x}_j\partial \dot{x}_m}\left(\frac{\partial g_m}{\partial x_n} + \frac{\partial a_m}{\partial x_n}\right)\frac{\partial p_{rn}}{\partial \dot{x}_k}\frac{\partial \dot{x}_j}{\partial \dot{x}_{Il}} + \frac{\partial p_{ri}}{\partial \dot{x}_m}\left[\left(\frac{\partial^2 g_m}{\partial x_s\partial x_n} + \frac{\partial^2 a_m}{\partial x_s\partial x_n}\right)\frac{\partial p_{rs}}{\partial \dot{x}_j}\frac{\partial \dot{x}_j}{\partial \dot{x}_{Il}} + \frac{\partial^2 a_m}{\partial \dot{x}_j\partial x_n}\frac{\partial \dot{x}_j}{\partial \dot{x}_{Il}}\right]\frac{\partial p_{rn}}{\partial \dot{x}_k} + \frac{\partial p_{ri}}{\partial \dot{x}_m}\left(\frac{\partial g_m}{\partial x_n} + \frac{\partial a_m}{\partial x_n}\right)\frac{\partial^2 p_{rn}}{\partial \dot{x}_j\partial \dot{x}_k}\frac{\partial \dot{x}_j}{\partial \dot{x}_{Il}} + \frac{\partial^2 p_{ri}}{\partial \dot{x}_j\partial \dot{x}_m}\frac{\partial a_m}{\partial \dot{x}_k}\frac{\partial \dot{x}_j}{\partial \dot{x}_{Il}} + \frac{\partial p_{ri}}{\partial \dot{x}_m}\left(\frac{\partial^2 a_m}{\partial x_s\partial \dot{x}_k}\frac{\partial p_{rs}}{\partial \dot{x}_j}\frac{\partial \dot{x}_j}{\partial \dot{x}_{Il}} + \frac{\partial^2 a_m}{\partial \dot{x}_j\partial \dot{x}_k}\frac{\partial \dot{x}_j}{\partial \dot{x}_{Il}}\right) = 0$$

If we change some dummy summation indices, we may factor out $\partial\dot{x}_j/\partial\dot{x}_{Il}$ so we have an expression of the form

$$(A_{jk}{}^i)(\partial \dot{x}_j/\partial \dot{x}_{Il}) = 0 \quad (22)$$

The assumed nonsingularity of $\partial\dot{x}/\partial\dot{x}_I$ is equivalent to its rows being linearly independent. Hence, for each fixed combination of i and k, Eq. (22) can be true if, and only if, the coefficients of each row vanish. Thus we finally have

$$\frac{d}{dt}\left(\frac{\partial^2 p_{ri}}{\partial \dot{x}_j\partial \dot{x}_k}\right) + \frac{\partial^2 p_{ri}}{\partial \dot{x}_m\partial \dot{x}_k}\left[\left(\frac{\partial g_m}{\partial x_n} + \frac{\partial a_m}{\partial x_n}\right)\frac{\partial p_{rn}}{\partial \dot{x}_j} + \frac{\partial a_m}{\partial \dot{x}_j}\right] + \frac{\partial^2 p_{ri}}{\partial \dot{x}_j\partial \dot{x}_m}\left(\frac{\partial g_m}{\partial x_n} + \frac{\partial a_m}{\partial x_n}\right)\frac{\partial p_{rn}}{\partial \dot{x}_k} + \frac{\partial p_{ri}}{\partial \dot{x}_m}\left[\left(\frac{\partial^2 g_m}{\partial x_s\partial x_n} + \frac{\partial^2 a_m}{\partial x_s\partial x_n}\right)\frac{\partial p_{rs}}{\partial \dot{x}_j} + \frac{\partial^2 a_m}{\partial \dot{x}_j\partial x_n}\right]\frac{\partial p_{rn}}{\partial \dot{x}_k} + \frac{\partial p_{ri}}{\partial \dot{x}_m}\left(\frac{\partial g_m}{\partial x_n} + \frac{\partial a_m}{\partial x_n}\right)\frac{\partial^2 p_{rn}}{\partial \dot{x}_j\partial \dot{x}_k} + \frac{\partial^2 p_{ri}}{\partial \dot{x}_j\partial \dot{x}_m}\frac{\partial a_m}{\partial \dot{x}_k} + \frac{\partial p_{ri}}{\partial \dot{x}_m}\left(\frac{\partial^2 a_m}{\partial x_n\partial \dot{x}_k}\frac{\partial p_{rn}}{\partial \dot{x}_j} + \frac{\partial^2 a_m}{\partial \dot{x}_j\partial \dot{x}_k}\right) = 0 \quad (23)$$

Now from Eqs. (4) and (7) we see that

$$\frac{\partial^2 x_i}{\partial \dot{x}_{Ij}\partial \dot{x}_{Ik}} = \frac{\partial^2 p_{ri}}{\partial \dot{x}_m\partial \dot{x}_n}\frac{\partial \dot{x}_m}{\partial \dot{x}_{Ij}}\frac{\partial \dot{x}_n}{\partial \dot{x}_{Ik}} + \frac{\partial p_{ri}}{\partial \dot{x}_n}\frac{\partial^2 \dot{x}_n}{\partial \dot{x}_{Ij}\partial \dot{x}_{Ik}}$$

from which it is not difficult to deduce that at $t = t_I'$ we have $\partial^2 p_{ri}/\partial\dot{x}_j\partial\dot{x}_k = 0$, since $\partial^2 x_i/\partial\dot{x}_{Ij}\partial\dot{x}_{Ik}$ and $\partial^2\dot{x}_n/\partial\dot{x}_{Ij}\partial\dot{x}_{Ik}$ are all zero at $t = t_I'$. This latter fact is true because the initial condition for the differential equations (the second-order variational equations), that the second-order partial derivatives of any component of position or velocity with respect to any combination of components of initial position or velocity obey, are zero.

To derive the differential equations that the second-order mixed partials of position-required with respect to velocity $\dot{x}$ and target position x_I obey, we write Eq. (13) in coordinate form as

$$\frac{d}{dt}\left(\frac{\partial p_{ri}}{\partial x_{Ik}}\right) + \frac{\partial p_{ri}}{\partial \dot{x}_m}\left(\frac{\partial g_m}{\partial x_n} + \frac{\partial a_m}{\partial x_n}\right)\frac{\partial p_{rn}}{\partial x_{Ik}} = 0 \quad (24)$$

Differentiating with respect to $\dot{x}_{Il}$, we have

$$\frac{d}{dt}\left(\frac{\partial^2 p_{ri}}{\partial \dot{x}_{Il}\partial x_{Ik}}\right) + \frac{\partial}{\partial \dot{x}_{Il}}\left[\frac{\partial p_{ri}}{\partial \dot{x}_m}\left(\frac{\partial g_m}{\partial x_n} + \frac{\partial a_m}{\partial x_n}\right)\frac{\partial p_{rn}}{\partial x_{Ik}}\right] = 0$$

which expands as Eq. (20) expanded to Eq. (21). Noting from the right-hand side of Eq. (4) that $\partial^2 p_{ri}/\partial\dot{x}_{Il}\partial x_{Ik} = (\partial^2 p_{ri}/\partial\dot{x}_j\partial x_{Ik})(\partial\dot{x}_j/\partial\dot{x}_{Il})$, it is possible to derive the desired result using precisely the same steps as the derivation of Eq. (23) from Eq. (21). The only differences are that fewer terms are involved, since Eq. (24) contains fewer terms than Eq.

(20), and one must substitute ∂x_{Ik} for $\partial \dot{x}_k$ in Eqs. (21–23). The result is

$$\frac{d}{dt}\left(\frac{\partial^2 p_{ri}}{\partial \dot{x}_j \partial x_{Ik}}\right) + \frac{\partial^2 p_{ri}}{\partial \dot{x}_m \partial x_{Ik}}\left[\left(\frac{\partial g_m}{\partial x_n} + \frac{\partial a_m}{\partial x_n}\right) \times \frac{\partial p_{rn}}{\partial \dot{x}_j} + \frac{\partial a_m}{\partial \dot{x}_j}\right] + \frac{\partial^2 p_{ri}}{\partial \dot{x}_j \partial \dot{x}_m}\left(\frac{\partial g_m}{\partial x_n} + \frac{\partial a_m}{\partial x_n}\right)\frac{\partial p_{rn}}{\partial x_{Ik}} + \frac{\partial p_{ri}}{\partial \dot{x}_m}\left[\left(\frac{\partial^2 g_m}{\partial x_s \partial x_n} + \frac{\partial^2 a_m}{\partial x_s \partial x_n}\right)\frac{\partial p_{rs}}{\partial \dot{x}_j} + \frac{\partial^2 a_m}{\partial \dot{x}_j \partial x_n}\right]\frac{\partial p_{rn}}{\partial x_{Ik}} + \frac{\partial p_{ri}}{\partial \dot{x}_m}\left(\frac{\partial g_m}{\partial x_n} + \frac{\partial a_m}{\partial x_n}\right)\frac{\partial^2 p_{rn}}{\partial \dot{x}_j \partial x_{Ik}} = 0 \quad (25)$$

By differentiating the coordinate form of Eq. (7) with respect to x_{Ik}, it is easy to see that the $\partial^2 p_{ri}/\partial \dot{x}_j \partial x_{Ik}$ are zero at $t = t_I'$.

To derive the differential equations that second-order partials of position-required with respect to target position obey, we differentiate Eq. (24) with respect to x_{Ij} to obtain

$$\frac{d}{dt}\left[\frac{d}{dx_{Ij}}\left(\frac{\partial p_{ri}}{\partial x_{Ik}}\right)\right] + \frac{d}{dx_{Ij}}\left[\frac{\partial p_{ri}}{\partial \dot{x}_m}\left(\frac{\partial g_m}{\partial x_n} + \frac{\partial a_m}{\partial x_n}\right)\frac{\partial p_{rn}}{\partial x_{Ik}}\right] = 0 \quad (26)$$

Noting from the right-hand side of Eq. (4) that

$$(d/dx_{Ij})\,(\partial p_{ri}/\partial x_{Ik}) = (\partial^2 p_{ri}/\partial \dot{x}_l \partial x_{Ik})\partial \dot{x}_l/\partial x_{Ij} + \partial^2 p_{ri}/\partial x_{Ij}\partial x_{Ik}$$

it is possible to expand Eq. (26), substitute from Eq. (9) and then Eq. (12), and combine terms similar to the previous derivations to obtain an expression of the form $(B_{lk}{}^i)(\partial \dot{x}_l/\partial x_{Ij}) + C_{jk}{}^i = 0$. For each combination of i, l, k, the $B_{lk}{}^i$ are precisely the left-hand side of Eq. (25) (with l in place of j) and, hence, zero. We are left with $C_{jk}{}^i = 0$ and this expression is

$$\frac{d}{dt}\left(\frac{\partial^2 p_{ri}}{\partial x_{Ij}\partial x_{Ik}}\right) + \frac{\partial^2 p_{ri}}{\partial \dot{x}_m \partial x_{Ik}}\left(\frac{\partial g_m}{\partial x_n} + \frac{\partial a_m}{\partial x_n}\right)\frac{\partial p_{rn}}{\partial x_{Ij}} + \frac{\partial^2 p_{ri}}{\partial x_{Ij}\partial \dot{x}_m}\left(\frac{\partial g_m}{\partial x_n} + \frac{\partial a_m}{\partial x_n}\right)\frac{\partial p_{rn}}{\partial x_{Ik}} + \frac{\partial p_{ri}}{\partial \dot{x}_m}\left(\frac{\partial^2 g_m}{\partial x_s \partial x_n} + \frac{\partial^2 a_m}{\partial x_s \partial x_n}\right)\frac{\partial p_{rs}}{\partial x_{Ij}}\frac{\partial p_{rn}}{\partial x_{Ik}} + \frac{\partial p_{ri}}{\partial \dot{x}_m}\left(\frac{\partial g_m}{\partial x_n} + \frac{\partial a_m}{\partial x_n}\right)\frac{\partial^2 p_{rn}}{\partial x_{Ij}\partial x_{Ik}} = 0 \quad (27)$$

By differentiating the coordinate form of Eq. (12) with respect to x_{Ik}, it is obvious that the $\partial^2 p_{ri}/\partial x_{Ij}\partial x_{Ik}$ are zero at $t = t_I'$.

Returning to Eq. (17), we note that the equation holds for $(x_I, \dot{x}_I, t_I)$ in a neighborhood of $(x_I', \dot{x}_I', t_I')$ so that $\partial p_{ri}/\partial t_I = -(\partial p_{ri}/\partial x_{Ik})\dot{x}_{Ik}$ holds in this neighborhood. Hence, differentiating Eq. (17) with respect to $\dot{x}_{Il}$, using the right-hand side of Eq. (4) and the nonsingularity of $\partial \dot{x}/\partial \dot{x}_I$, and evaluating at the reference point give

$$\partial^2 p_{ri}/\partial \dot{x}_j \partial t_I = -(\partial^2 p_{ri}/\partial \dot{x}_j \partial x_{Ik})\dot{x}_{Ik}' \quad (28)$$

Now taking the total derivative of Eq. (17) with respect to x_{Ij} and t_I, using the right-hand side of Eq. (4) and Eq. (28), and evaluating at the reference point give

$$\partial^2 p_{ri}/\partial x_{Ij}\partial t_I = -(\partial^2 p_{ri}/\partial x_{Ij}\partial x_{Ik})\dot{x}_{Ik}' \quad (29)$$

$$\partial^2 p_{ri}/\partial t_I{}^2 = -(\partial^2 p_{ri}/\partial t_I \partial x_{Ik})\dot{x}_{Ik}' \quad (30)$$

In the case where variations in x_I are not arbitrary but are constrained to satisfy $x_T(t_I)$, we do not want to solve Eqs. (25) and (27) for all values of j and k. To avoid this, we write out the second-order Taylor series approximation to Δp_{ri}, just as we did in the first-order case [Eq. (18)], let $x_T(t_I) - x_T(t_I')$ be approximated by $[\dot{x}_T(t_I')]\Delta t_I$, and combine terms to obtain

$$\Delta p_{ri} \approx \text{Linear terms} + \frac{1}{2}\left\{\frac{\partial^2 p_{ri}}{\partial \dot{x}_j \partial \dot{x}_k}\Delta \dot{x}_j \Delta \dot{x}_k + 2\left[\frac{\partial^2 p_{ri}}{\partial \dot{x}_j \partial t_I} + \frac{\partial^2 p_{ri}}{\partial \dot{x}_j \partial x_{Ik}}\dot{x}_{Tk}(t_I')\right] \times \Delta \dot{x}_j \Delta t_I + \left[\frac{\partial^2 p_{ri}}{\partial x_{Ij}\partial x_{Ik}}\dot{x}_{Tk}(t_I')\dot{x}_{Tj}(t_I') + 2\frac{\partial^2 p_{ri}}{\partial x_{Ij}\partial t_I}\dot{x}_{Tj}(t_I') + \frac{\partial^2 p_{ri}}{\partial t_I{}^2}\right]\Delta t_I{}^2\right\} \quad (31)$$

Of course, we calculate the $\partial^2 p_{ri}/\partial \dot{x}_j \partial \dot{x}_k$ by solving Eq. (23). Now defining $u_j{}^i$ to be the coefficient of $\Delta \dot{x}_j \Delta t_I$ in Eq. (31) and then using Eq. (28) we have $u_j{}^i = (\partial^2 p_{ri}/\partial \dot{x}_j \partial x_{Ik})[\dot{x}_{Tk}(t_I') - \dot{x}_{Ik}']$. Noting that $d[\dot{x}_{Tk}(t_I') - \dot{x}_{Ik}']/dt = 0$, we multiply Eq. (25) by $[\dot{x}_{Tk}(t_I') - \dot{x}_{Ik}']$, define $v^n \equiv (\partial p_{rn}/\partial x_{Ik})[\dot{x}_{Tk}(t_I') - \dot{x}_{Ik}']$, and sum on k to obtain

$$\frac{du_j{}^i}{dt} + u_m{}^i\left[\left(\frac{\partial g_m}{\partial x_n} + \frac{\partial a_m}{\partial x_n}\right)\frac{\partial p_{rn}}{\partial \dot{x}_j} + \frac{\partial a_m}{\partial \dot{x}_j}\right] + \frac{\partial^2 p_{ri}}{\partial \dot{x}_j \partial \dot{x}_m}\left(\frac{\partial g_m}{\partial x_n} + \frac{\partial a_m}{\partial x_n}\right)v^n + \frac{\partial p_{ri}}{\partial \dot{x}_m}\left[\left(\frac{\partial^2 g_m}{\partial x_s \partial x_n} + \frac{\partial^2 a_m}{\partial x_s \partial x_n}\right)\frac{\partial p_{rs}}{\partial \dot{x}_j} + \frac{\partial^2 a_m}{\partial \dot{x}_j \partial x_n}\right]v^n + \frac{\partial p_{ri}}{\partial \dot{x}_m}\left(\frac{\partial g_m}{\partial x_n} + \frac{\partial a_m}{\partial x_n}\right)u_j{}^n = 0 \quad (32)$$

Observe that v^n is the coefficient of Δt_I in Eq. (19) [using Eq. (17)] and, hence, obeys Eq. (14) with initial condition at the target $\dot{x}_T(t_I') - \dot{x}_I'$. Of course, the initial conditions for Eq. (32) at the target are zero since the $\partial^2 p_{ri}/\partial \dot{x}_j \partial x_{Ik}$ are all zero there.

Now the coefficient of $\Delta t_I{}^2$ in Eq. (31), call it w^i, can be written using Eqs. (29) and (30) as

$$(\partial^2 p_{ri}/\partial x_{Ij}\partial x_{Ik})[\dot{x}_{Tj}(t_I')\dot{x}_{Tk}(t_I') - 2\dot{x}_{Ik}'\dot{x}_{Tj}(t_I') + \dot{x}_{Ik}'\dot{x}_{Ij}']$$

so by multiplying Eq. (27) by $[\dot{x}_{Tk}(t_I') - \dot{x}_{Ik}'][\dot{x}_{Tj}(t_I') - \dot{x}_{Ij}']$ and summing on j and k give

$$\frac{dw^i}{dt} + 2u_m{}^i\left(\frac{\partial g_m}{\partial x_n} + \frac{\partial a_m}{\partial x_n}\right)v^n + \frac{\partial p_{ri}}{\partial \dot{x}_m}\left(\frac{\partial^2 g_m}{\partial x_s \partial x_n} + \frac{\partial^2 a_m}{\partial x_s \partial x_n}\right)v^s v^n + \frac{\partial p_{ri}}{\partial \dot{x}_m}\left(\frac{\partial g_m}{\partial x_n} + \frac{\partial a_m}{\partial x_n}\right)w^n = 0 \quad (33)$$

Of course, the initial condition for Eq. (33) at the target is zero. Hence, to obtain the first- and second-order Taylor series coefficients for p_r, where variations in x_I are constrained to be $x_T(t_I)$ and variations in $x_T(t_I)$ are approximated by $[\dot{x}_T(t_I')]\Delta t_I$, we solve Eqs. (1, 10, 14, 23, 32, and 33). If the linear approximation for variations of $x_T(t_I)$ is inadequate, we use $[\dot{x}_T(t_I')]\Delta t_I + [\ddot{x}_T(t_I')]\Delta t_I{}^2/2$. The reader is invited to work out the appropriate differential equations and initial conditions in this case.

Relationships between Derivatives

In this section algebraic relationships between the first- and second-order derivatives of velocity-required and position-required are derived. The results are Eqs. (40–43, and 47–50). Furthermore, we show how to derive differential equations for the partial derivatives of velocity-required by differentiating these algebraic relationships with respect to t and substituting from the differential equations of the appropriate partial derivatives of position-required.

Since the desired relationships are most easily derived in complete generality, we suppose that f is an n-dimensional vector function such that $f(u_0,z_0,y_0) = 0$ for some fixed point (u_0,z_0,y_0) in $(2n + p)$-dimensional Euclidean space, and we suppose that f possesses continuous second-order partial derivatives in a neighborhood of this point. The vectors u, z, and y are of dimension p, n, and n respectively. If the inverse of $\partial f/\partial y$ exists at the fixed point, then the Implicit Function Theorem states that y is a unique function ϕ of, and possesses continuous second-order partial derivatives with respect to, u and z in a neighborhood of (u_0,z_0) such that $f[u,z,\phi(u,z)] = 0$ in this neighborhood. Moreover, $\phi(u_0,z_0) = y_0$. It follows that

$$0 = \partial f/\partial z + (\partial f/\partial y)(\partial\phi/\partial z) \tag{34}$$

$$0 = \partial f/\partial u + (\partial f/\partial y)(\partial\phi/\partial u) \tag{35}$$

in the above neighborhood. Similarly, if the inverse of $\partial f/\partial z$ exists at the fixed point, then we have z as a function θ of u and y such that $f[u,\theta(u,y),y] = 0$ and $\theta(u_0,y_0) = z_0$, from which it follows that

$$0 = \partial f/\partial y + (\partial f/\partial z)(\partial\theta/\partial y) \tag{36}$$

$$0 = \partial f/\partial u + (\partial f/\partial z)(\partial\theta/\partial u) \tag{37}$$

Now solving for $\partial\phi/\partial z$ in Eq. (34) and $\partial\theta/\partial y$ in Eq. (36) and comparing results shows that

$$\partial\phi/\partial z = (\partial\theta/\partial y)^{-1} \tag{38}$$

Solving for $\partial\phi/\partial u$ in Eq. (35), $\partial\theta/\partial u$ in Eq. (37), and inserting $(\partial f/\partial z)(\partial f/\partial z)^{-1}$ give

$$\partial\phi/\partial r = -(\partial\phi/\partial z)(\partial\theta/\partial u) \tag{39}$$

In terms of velocity and position-required, Eqs. (38) and (39) become

$$\partial v_r/\partial x = (\partial p_r/\partial\dot{x})^{-1} \tag{40}$$

$$\partial v_r/\partial x_I = -(\partial v_r/\partial x)(\partial p_r/\partial x_I) \tag{41}$$

$$\partial v_r/\partial t_I = -(\partial v_r/\partial x)(\partial p_r/\partial t_I) \tag{42}$$

The derivations of Eqs. (41) and (42) assume variations in x_I and t_I are independent. In the instance where variations in x_I are not arbitrary but are constrained to be $x_T(t_I)$, it is easy to see that

$$\frac{\partial v_r}{\partial x_I}\dot{x}_T(t_I') + \frac{\partial v_r}{\partial t_I} = -\frac{\partial v_r}{\partial x}\left[\frac{\partial p_r}{\partial x_I}\dot{x}_T(t_I') + \frac{\partial p_r}{\partial t_I}\right] \tag{43}$$

which is the coefficient of Δt_I in the linear Taylor series approximation for Δv_r [see Eqs. (18) and (19)].

Noting that $(\partial v_r/\partial x)(\partial p_r/\partial\dot{x}) = I$ in a region where $\partial v_r/\partial x$ exists, we differentiate this expression with respect to t and then substitute Eq. (10) for $d(\partial p_r/\partial\dot{x})/dt$ and use Eq. (40) to obtain the differential equation for $\partial v_r/\partial x$. Similarly, differentiating Eqs. (41–43) with respect to t and making appropriate substitutions gives differential equations for $\partial v_r/\partial x_I$, $\partial v_r/\partial t_I$, etc.

To derive the desired second-order relationships efficiently, we observe that θ and ϕ are inverse functions of one another for every fixed u in a neighborhood of u_0.§ This follows from the fact that $\partial\phi/\partial z$ is nonsingular in a neighborhood of (u_0,z_0) so that ϕ^{-1} exists and has derivative $\partial\phi^{-1}/\partial y = (\partial\phi/\partial z)^{-1} = \partial\theta/\partial y$.[10] Since this relation holds in a neighborhood of (u_0,y_0) and $\phi^{-1}(u_0,y_0) = z_0 = \theta(u_0,y_0)$, it follows from a first-order Taylor's formula with integral remainder[11] that $\phi^{-1} = \theta$ in a neighborhood of (u_0,y_0). Hence,

$$y = H(u,y) \equiv \phi[u,\theta(u,y)] \tag{44}$$

§ This basic observation was made by P. Cefola and results in significant reductions in the algebraic manipulations required to derive the desired second-order relationships. The original derivation, which used the Implicit Function Theorem, was much more lengthy.[6]

From this equation, we can easily derive the general relation

$$I = (\partial H/\partial y)\{(u_0,y_0)\} = [(\partial\phi/\partial z)\{u_0,\theta(u_0,y_0)\}][(\partial\theta/\partial y)\{(u_0,y_0)\}] \tag{45}$$

In Eq. (45) we have deliberately included the fact that the relation involving the derivative of a composition of functions, which we usually write as $(\partial\phi/\partial z)(\partial\theta/\partial y)$, is really only true when these derivatives are evaluated at the appropriately corresponding points. Hence, they depend upon, i.e., are functions of, u_0 and y_0. Since the relation in Eq. (45) holds true for (u,y) in a neighborhood of (u_0,y_0), we may differentiate the coordinate form of Eq. (45) with respect to y_l to obtain¶

$$0 = \partial^2 H_i/\partial y_l\partial y_m = (\partial/\partial y_l)[(\partial\phi_i/\partial z_h)\partial\theta_h/\partial y_m] = (\partial^2\phi_i/\partial z_s\partial z_h)(\partial\theta_s/\partial y_l)\partial\theta_h/\partial y_m + (\partial\phi_i/\partial z_h)\partial^2\theta_h/\partial y_l\partial y_m$$

Of course, this relation depends upon evaluating the derivatives at the appropriately corresponding points, but we have now suppressed this dependence as is customary. Multiplying by $(\partial\phi_l/\partial z_j)(\partial\phi_m/\partial z_k)$, summing on l and m, using Eq. (38), and rearranging give

$$\partial^2\phi_i/\partial z_j\partial z_k = -(\partial\phi_i/\partial z_h)(\partial^2\theta_h/\partial y_l\partial y_m)(\partial\phi_l/\partial z_j)\partial\phi_m/\partial z_k \tag{46}$$

Hence, if we know the first- and second-order partial derivatives of the θ's with respect to the y's, we can calculate the first- and second-order partial derivatives of the ϕ's with respect to the z's using Eqs. (38) and (46). Doubtlessly, formulas for the relationships between even higher-order partial derivatives can be derived. In terms of velocity and position-required, Eq. (46) becomes

$$\frac{\partial^2 v_{ri}}{\partial x_j\partial x_k} = -\frac{\partial v_{ri}}{\partial x_h}\frac{\partial^2 p_{rh}}{\partial\dot{x}_l\partial\dot{x}_m}\frac{\partial v_{rl}}{\partial x_j}\frac{\partial v_{rm}}{\partial x_k} \tag{47}$$

Differentiating Eq. (47) with respect to t and substituting from Eq. (23) and the differential equation for $\partial v_r/\partial x$ give differential equations for the $\partial^2 v_{ri}/\partial x_j\partial x_k$.

Now differentiating Eq. (45) with respect to u_j yields

$$0 = \partial^2 H_i/\partial u_j\partial y_l = (\partial/\partial u_j)[(\partial\phi_i/\partial z_h)\partial\theta_h/\partial y_l] = \left(\frac{\partial^2\phi_i}{\partial u_j\partial z_h} + \frac{\partial^2\phi_i}{\partial z_m\partial z_h}\frac{\partial\theta_m}{\partial u_j}\right)\frac{\partial\theta_h}{\partial y_l} + \frac{\partial\phi_i}{\partial z_h}\frac{\partial^2\theta_h}{\partial u_j\partial y_l}$$

Multiplying by $\partial\phi_l/\partial z_k$, summing on l, using Eq. (38), changing one dummy summation index, and rearranging yield

$$\frac{\partial^2\phi_i}{\partial u_j\partial z_k} = -\frac{\partial^2\phi_i}{\partial z_h\partial z_k}\frac{\partial\theta_h}{\partial u_j} - \frac{\partial\phi_i}{\partial z_h}\frac{\partial^2\theta_h}{\partial u_j\partial y_l}\frac{\partial\phi_l}{\partial z_k}$$

In terms of velocity and position-required, this equation becomes

$$\frac{\partial^2 v_{ri}}{\partial x_{Ij}\partial x_k} = -\frac{\partial^2 v_{ri}}{\partial x_h\partial x_k}\frac{\partial p_{rh}}{\partial x_{Ij}} - \frac{\partial v_{ri}}{\partial x_h}\frac{\partial^2 p_{rh}}{\partial x_{Ij}\partial\dot{x}_l}\frac{\partial v_{rl}}{\partial x_k} \tag{48}$$

$$\frac{\partial^2 v_{ri}}{\partial t_I\partial x_k} = -\frac{\partial^2 v_{ri}}{\partial x_h\partial x_k}\frac{\partial p_{rh}}{\partial t_I} - \frac{\partial v_{ri}}{\partial x_h}\frac{\partial^2 p_{rh}}{\partial t_I\partial\dot{x}_l}\frac{\partial v_{rl}}{\partial x_k} \tag{49}$$

Of course, these algebraic equations may also be differentiated with respect to t to obtain differential equations for the appropriate partial derivatives of velocity-required.

Finally, from Eq. (44) we obtain

$$0 = (\partial H/\partial u)\{u_0,y_0\} = (\partial\phi/\partial u)\{u_0,\theta(u_0,y_0)\} + [(\partial\phi/\partial z)\{u_0,\theta(u_0,y_0)\}][(\partial\theta/\partial u)\{u_0,y_0\}]$$

¶ In the subsequent derivations of this section, the indices for ϕ, θ, y, and z vary over $1,\ldots,n$; the indices for u vary over $1,\ldots,p$; and the indices of p_r, v_r, x, $\dot{x}$, and x_I vary over 1,2,3.

so that differentiating the coordinate form of this relation with respect to u_j gives

$$0 = \frac{\partial^2 H_i}{\partial u_j \partial u_k} = \frac{\partial^2 \phi_i}{\partial u_j \partial u_k} + \frac{\partial^2 \phi_i}{\partial z_h \partial u_k}\frac{\partial \theta_h}{\partial u_j} + \left(\frac{\partial^2 \phi_i}{\partial u_j \partial z_h} + \frac{\partial^2 \phi_i}{\partial z_l \partial z_h}\frac{\partial \theta_l}{\partial u_j}\right)\frac{\partial \theta_h}{\partial u_k} + \frac{\partial \phi_i}{\partial z_h}\frac{\partial^2 \theta_h}{\partial u_j \partial u_k}$$

Solving for $\partial^2\phi_i/\partial u_j \partial u_k$ gives

$$\frac{\partial^2 \phi_i}{\partial u_j \partial u_k} = -\frac{\partial^2 \phi_i}{\partial z_h \partial u_k}\frac{\partial \theta_h}{\partial u_j} - \left(\frac{\partial^2 \phi_i}{\partial u_j \partial z_h} + \frac{\partial^2 \phi_i}{\partial z_l \partial z_h}\frac{\partial \theta_l}{\partial u_j}\right)\frac{\partial \theta_h}{\partial u_k} - \frac{\partial \phi_i}{\partial z_h}\frac{\partial^2 \theta_h}{\partial u_j \partial u_k} \quad (50)$$

The reader should be able by now to write out the expressions for $\partial^2 v_{ri}/\partial x_{Ij}\partial x_{Ik}$, $\partial^2 v_{ri}/\partial x_{Ij}\partial t_I$, and $\partial^2 v_{ri}/\partial t_I^2$ from Eq. (50). In the instance where variations in x_I are not arbitrary but are constrained to be $x_T(t_I)$, it is possible to derive relationships similar to Eq. (43) for second-order coefficients. Again, the reader is invited to do so.

The partial derivatives $\partial^2 v_{ri}/\partial t \partial x_j$, etc., involving time, can be computed algebraically using other first- and second-order partial derivatives already computed [Ref. 1, Eq. (11)].

Appendix

All notations in this Appendix follow the notations in the beginning of the section entitled "First-Order Derivatives of Position-Required." Suppose $\partial x/\partial x_0$ is nonsingular when calculated from t_0' to t_I' along the reference trajectory defined by $(x_0',\dot{x}_0',t_0')$. Then the Implicit Function Theorem tells us that open sets U_1,U_2,U_3, and U_4 exist containing $\dot{x}_0',t_0',t_I'$, and x_I' such that p_r is a unique, continuous function on the open set $U \equiv U_1 \mathbf{X} U_2 \mathbf{X} U_3 \times U_4$, possessing continuous second-order derivatives on U, and satisfying

$$x_I = x[p_r(\dot{x}_0,t_0,t_I,x_I),\dot{x}_0,t_0,t_I] \quad (51)$$

whenever $(\dot{x}_0,t_0,t_I,x_I)$ is in U. The uniqueness of p_r means that $p_r(\dot{x}_0,t_0,t_I,x_I)$ is the only initial position satisfying Eq. (51) for $(\dot{x}_0,t_0,t_I,x_I)$ in U.

On the other hand, open sets U_1',U_2',U_3', and U_4' exist containing $x_I',\dot{x}_I',t_I'$, and t_0' such that $\dot{x}(x_I,\dot{x}_I,t_I,t_0)$ is in U_1 whenever $(x_I,\dot{x}_I,t_I,t_0)$ is in the open set $U_1'\mathbf{X}U_2'\mathbf{X}U_3'\times U_4'$. This follows from the continuity of solutions of Eq. (1) with respect to initial conditions and time (Ref. 7, p. 22). Letting $V_1 \equiv U_1'U_4$, $V_2 \equiv U_2'$, $V_3 \equiv U_3' \cap U_3$, and $V_4 \equiv U_4' \cap U_2$, we may conclude that $[\dot{x}(x_I,\dot{x}_I,t_I,t_0), t_0,t_I,x_I]$ is in U whenever $(x_I,\dot{x}_I,t_I,t_0)$ is in the open set $V \equiv V_1\mathbf{X}V_2\mathbf{X}V_3\times V_4$.

Finally, we note that the solutions to Eq. (1) satisfying specific conditions $(x_2,\dot{x}_2,t_2)$ are unique. This follows from our virtually global assumption on the existence of continuous second-order derivatives of g and a (Ref. 7, p. 12). Solutions

$$x(x_I,\dot{x}_I,t_I,t),\ \dot{x}(x_I,\dot{x}_I,t_I,t) \quad (52)$$

to Eq. (1) starting at $(x_I,\dot{x}_I,t_I)$ satisfy $x_I = x(x_I,\dot{x}_I,t_I,t_I)$ and $\dot{x}_I = \dot{x}(x_I,\dot{x}_I,t_I,t_I)$. Hence, the solutions to Eq. (1)

$$\begin{aligned} &x[x(x_I,\dot{x}_I,t_I,t_0),\ \dot{x}(x_I,\dot{x}_I,t_I,t_0),t_0,t] \\ &\dot{x}[x(x_I,\dot{x}_I,t_I,t_0),\ \dot{x}(x_I,\dot{x}_I,t_I,t_0),t_0,t] \end{aligned} \quad (53)$$

starting at $x(x_I,\dot{x}_I,t_I,t_0)$, $\dot{x}(x_I,\dot{x}_I,t_I,t_0)$, and t_0 must satisfy

$$x_I = x[x(x_I,\dot{x}_I,t_I,t_0),\ \dot{x}(x_I,\dot{x}_I,t_I,t_0),t_0,t_I]$$

$$\dot{x}_I = \dot{x}[x(x_I,\dot{x}_I,t_I,t_0),\ \dot{x}(x_I,\dot{x}_I,t_I,t_0),t_0,t_I]$$

Otherwise, there would be two distinct solutions to Eq. (1) satisfying the conditions $x(x_I,\dot{x}_I,t_I,t_0)$, $\dot{x}(x_I,\dot{x}_I,t_I,t_0)$ and t_0, namely, Eqs. (52) and (53).

Now restricting $(x_I,\dot{x}_I,t_I,t_0)$ to be in V, we see that the following equations are both satisfied

$$x_I = x[x(x_I,\dot{x}_I,t_I,t_0),\ \dot{x}(x_I,\dot{x}_I,t_I,t_0),t_0,t_I]$$

$$x_I = x\{p_r[\dot{x}(x_I,\dot{x}_I,t_I,t_0),t_0,t_I,x_I],\ \dot{x}(x_I,\dot{x}_I,t_I,t_0),t_0,t_I\}$$

from which we may conclude that $x(x_I,\dot{x}_I,t_I,t_0) = p_r[\dot{x}(x_I,\dot{x}_I,t_I,t_0),t_0,t_I,x_I]$. Since this argument holds true whenever $\partial x/\partial x_0$ is nonsingular, we may conclude

$$x(x_I,\dot{x}_I,t_I,t) = p_r[\dot{x}(x_I,\dot{x}_I,t_I,t),t,t_I,x_I]$$

for every time t for which the nonsingularity condition holds.

A similar relationship can be derived for v_r in much the same manner as the one for p_r. This relationship can then be used to directly derive Riccati-type differential equations that the derivatives of v_r obey (much as the Riccati-type differential equations for the derivatives of p_r were derived) rather than the method suggested in the section entitled "Relationships between Derivatives."

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